Mengham Junior School

Progression from mental to written strategies for the calculation strand

Adapted from the Renewed Framework 'Guidance paper on calculation' by Linda Bastable and Anna Ward Oct 2007

Progression from mental to written strategies for Calculation

"Being numerate implies an ability to calculate **effectively** and **efficiently**."

Although formal written procedures are not introduced in the Renewed framework for Maths until late in Year 3 or early in Year 4, when they are developed it would enhance pupils' confidence if a common approach within the school was pursued across the key stages.

Many pupils have difficulty with the procedures if they are taught in isolation and not related back to earlier experiences. When errors occur it is important to be able to go back to an earlier stage where understanding was secure leading to accurate use of a strategy.

Depending on the numerical problem being solved, the following steps provide a hierarchy of approach:

Do I work it out

In my head;

With **jottings**;

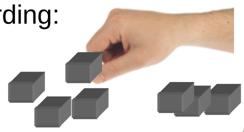
With a **procedure**;

With a **calculator.**

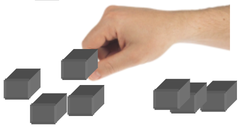
At all stages learners should be encouraged to use the approach that the feel most comfortable with at the time.

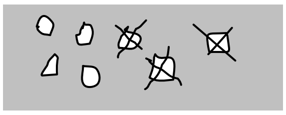
Steps of recording:

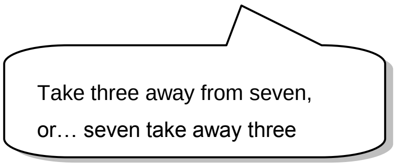
1) Apparatus



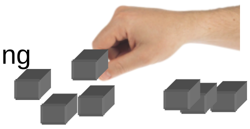
2) Apparatus with picture

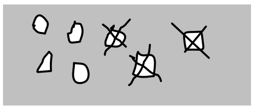


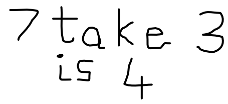




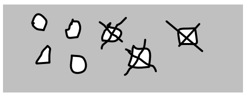
3) Apparatus with picture and recording

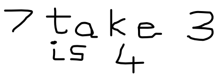




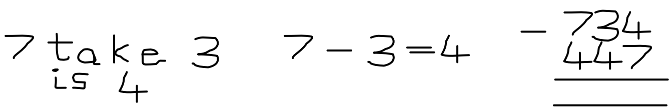


4) Picture with recording





5) Recording (with increasing efficiency and formality)



(Taken from: Towards a Common Approach by Jim Noonan for Portsmouth City Council 2005)

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An important concept necessary for calculation is an understanding of 'equality' (=) as a balance between 2 sides in an equation (a good model for this is a number balance, or 2 balance pans)

This progression document has been adapted from the 'Guidance document for Calculation' from the Renewed Framework for Mathematics.

This document has generally kept to the stages of calculation presented in the Renewed Framework Guidance, but has **been amended to add prior steps and the mental strategies leading to more formal written methods**. These changes further enhance pupil understanding of the underlying concepts of written calculation, and thus help them to progress to more efficient methods. Although a range of mental methods is to be encouraged, this document deals only with those strategies that lead to an understanding, and development, of standard efficient written methods.

WHEN ARE CHILDREN READY FOR FORMAL WRITTEN CALCULATIONS?

Addition and subtraction

- Do they know addition and subtraction facts to 20?
- Do they understand place value and can they partition numbers?
- Can they add three single digit numbers mentally?
- Can they add and subtract any pair of two digit numbers mentally?
- Can they explain their mental strategies orally and record them using informal jottings?

Multiplication and division

- Do they know the 2, 3, 4, 5 and 10 time table
- Do they know the result of multiplying by 0 and 1?
- Do they understand 0 as a place holder?
- Can they multiply two and three digit numbers by 10 and 100?
- Can they double and halve two digit numbers mentally?
- Can they use multiplication facts they know to derive mentally other multiplication facts that they do not know?
- Can they explain their mental strategies orally and record them using informal jottings?

The above lists are not exhaustive but are a guide for the teacher to judge when a child is ready to move from informal to formal methods of calculation.

When children leave primary school they:

- have a secure knowledge of number facts and a good understanding of the four operations;
- are able to use this knowledge and understanding to carry out calculations mentally and to apply general strategies when using one-digit and two-digit numbers and particular strategies to special cases involving bigger numbers;
- make use of diagrams and informal notes to help record steps and part answers when using mental methods that generate more information than can be kept in their heads;
- have an efficient, reliable, compact written method of calculation for each operation that children can apply with confidence when undertaking calculations that they cannot carry out mentally;
- use a calculator effectively, using their mental skills to monitor the process, check the steps involved and decide if the numbers displayed make sense.

The objectives in the renewed Framework show the progression in methods of calculation in the strands 'Using and Applying mathematics' and 'Calculation.

<table>
<tr><th>Using and applying mathematics</th><th>Calculating</th></tr>
<tr><td>FS
Describe solutions to practical problems, drawing on experience, talking about their own ideas, methods and choices</td><td>FS
Begin to relate addition to combining two groups of objects and subtraction to 'taking away'
In practical activities and discussion begin to use the vocabulary involved in adding and subtracting
Count repeated groups of the same size
Share objects into equal groups and count how many in each group</td></tr>
<tr><td>Year 1
Solve problems involving counting, adding, subtracting, doubling or halving in the context of numbers, measures or money, for example to 'pay' and 'give change'
Describe a puzzle or problem using numbers, practical materials and diagrams; use these to solve the problem and set the solution in the original context</td><td>Year 1
Relate addition to counting on; recognise that addition can be done in any order; use practical and informal written methods to support the addition of a one-digit number or a multiple of 10 to a one-digit or two-digit number
Understand subtraction as 'take away' and find a 'difference' by counting up; use practical and informal written methods to support the subtraction of a one-digit number from a one-digit or two-digit number and a multiple of 10 from a two-digit number
Use the vocabulary related to addition and subtraction and symbols to describe and record addition and subtraction number sentences</td></tr>
<tr><td>Year 2
Solve problems involving addition, subtraction, multiplication or division in contexts of numbers, measures or pounds and pence
Identify and record the information or calculation needed to solve a puzzle or problem; carry out the steps or calculations and check the solution in the context of the problem</td><td>Year 2
Represent repeated addition and arrays as multiplication, and sharing and repeated subtraction (grouping) as division; use practical and informal written methods and related vocabulary to support multiplication and division, including calculations with remainders
Use the symbols +, –, ×, ÷ and = to record and interpret number sentences involving all four operations; calculate the value of an unknown in a number sentence (e.g. ☐ ÷ 2 = 6, 30 – ☐ = 24)</td></tr>
<tr><td>Year 3
Solve one-step and two-step problems involving numbers, money or measures, including time, choosing and carrying out appropriate calculations
Represent the information in a puzzle or problem using numbers, images or diagrams; use these to find a solution and present it in context, where appropriate using £.p notation or units of measure</td><td>Year 3
Develop and use written methods to record, support or explain addition and subtraction of two-digit and three-digit numbers
Use practical and informal written methods to multiply and divide two-digit numbers (e.g. 13 × 3, 50 ÷ 4); round remainders up or down, depending on the context
Understand that division is the inverse of multiplication and vice versa; use this to derive and record related multiplication and division number sentences</td></tr>
</table>

Using and applying mathematics	Calculating
Year 4 • Solve one-step and two-step problems involving numbers, money or measures, including time; choose and carry out appropriate calculations, using calculator methods where appropriate • Represent a puzzle or problem using number sentences, statements or diagrams; use these to solve the problem; present and interpret the solution in the context of the problem	**Year 4** • Refine and use efficient written methods to add and subtract two-digit and three-digit whole numbers and £.p • Develop and use written methods to record, support and explain multiplication and division of two-digit numbers by a one-digit number, including division with remainders (e.g. 15 × 9, 98 ÷ 6)
Year 5 • Solve one-step and two-step problems involving whole numbers and decimals and all four operations, choosing and using appropriate calculation strategies, including calculator use • Represent a puzzle or problem by identifying and recording the information or calculations needed to solve it; find possible solutions and confirm them in the context of the problem	**Year 5** • Use efficient written methods to add and subtract whole numbers and decimals with up to two places • Use understanding of place value to multiply and divide whole numbers and decimals by 10, 100 or 1000 • Refine and use efficient written methods to multiply and divide HTU × U, TU × TU, U.t × U and HTU ÷ U
Year 6 • Solve multi-step problems, and problems involving fractions, decimals and percentages; choose and use appropriate calculation strategies at each stage, including calculator use • Represent and interpret sequences, patterns and relationships involving numbers and shapes; suggest and test hypotheses; construct and use simple expressions and formulae in words then symbols (e.g. the cost of *c* pens at 15 pence each is 15*c* pence)	**Year 6** • Use efficient written methods to add and subtract integers and decimals, to multiply and divide integers and decimals by a one-digit integer, and to multiply two-digit and three-digit integers by a two-digit integer

Calculation methods for addition

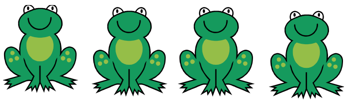
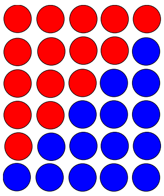
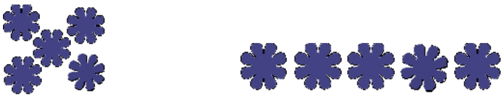
The aim is that children use mental methods when appropriate, but for calculations that they cannot do in their heads they use a quick and efficient written method accurately and with confidence.
Children are entitled to be taught and to acquire secure mental methods of calculation and an efficient written method of calculation for addition which they know they can rely on when mental methods are not appropriate. If, at any time, children find that the method they are using is badly understood and therefore leading to errors, they should return to the last step they did accurately and with understanding.

It is expected that most children will begin to record some calculations more formally by the end of Year 4 (i.e Stage 5)

To add successfully, children need to be able to:

- recall all addition pairs to 20 and number bonds to 10;
- add three or more one-digit numbers mentally, such as 5 + 8 + 4;
- add multiples of 10 (such as 60 + 70) or of 100 (such as 600 + 700) using the related addition fact, 6 + 7, and their knowledge of place value;
- understand place value - able to partition two-digit and three-digit numbers into multiples of 100, 10 and 1 in different ways.

Note: *It is important that children’s mental methods of calculation are practised and secured alongside their learning and use of an efficient written method for addition.*

Early steps: Combining 2 groups of objects (FS)	
• Early steps of calculation start with the following ‘pre’calculation’ steps: • 1:1 Correspondence	Each item is numbered as it is counted…first by touching and counting, then by seeing and ‘mentally touching’ The 4th item in a line relates to number ‘4’  1 2 3 4
• Knowing the ‘family of a number	 5 4 + 1 3 + 2 2 + 3 1 + 4 5 The family of 5. There are always 5 spots but 5 can be made in different ways.
• Conservation of number	If you rearrange a group of objects, changing the pattern doesn’t alter the number 

Stage 1: Drawing objects in a group

FS: Children can record what they have done by drawing or tallying

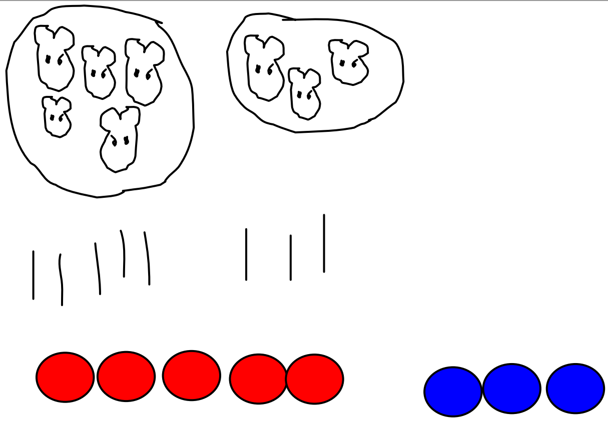
Stages in development:

1) Each set is counted out, the sets are combined and the resulting set is counted

2) The first set is put out and the second set is ‘counted on’

3) Materials are not needed

Model and encourage use of mathematical language, for example ‘count’, ‘count on’, ‘how many’, ‘altogether’, ‘add’, ‘one less ’ and ‘number before’



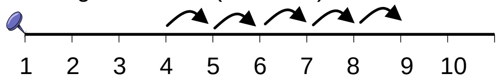
Stage 2: A marked and numbered number line (also bead bars, bead strings)

- Children extend counting on from Stage 1, to just thinking about the numbers involved. However, they may still need to use the horizontal number line to help them count or add on.
- At first, a totally marked and numbered line in ones (representing counting) is used. This then leads to using increasing steps with some missing numbers as children begin to use the line as an aid to mental counting.
- Numbered line in steps of 2, 5, 10, with other numbers added fairly accurately into the gaps as needed

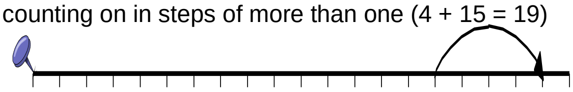
Steps in addition can be recorded on a number line.

Progression in use of number lines:

1) Fully marked and fully numbered number line – counting on in ones (4 + 5 = 9)



2) Fully marked and fully numbered number line – counting on in steps of more than one (4 + 15 = 19)



3) Fully marked and partially numbered number line – counting on in steps of more than one (20 + 15 = 35)

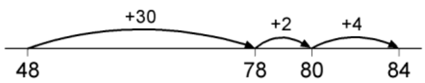
Stage 3: The empty number line

- The mental methods that lead to column addition generally involve partitioning, e.g. adding the tens and ones separately, often starting with the tens. Children need to be able to partition numbers in ways other than into tens and ones to help them make multiples of ten by adding in steps.
- The empty number line helps to record the steps on the way to calculating the total.

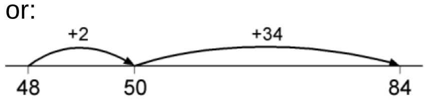
Steps in addition can be recorded on a number line. The steps often bridge through a multiple of 10.

8 + 7 = 15

48 + 36 = 84



or:



Amended from Renewed Framework Guidance

Stage 4: Partitioning

- The next stage is to record mental methods using partitioning. Add the tens and then the ones to form partial sums and then add these partial sums.
- Partitioning both numbers into tens and ones mirrors the column method where ones are placed under ones and tens under tens. This also links to mental methods.

Record steps in addition using partitioning:

$47 + 76 = 47 + 70 + 6 = 117 + 6 = 123$

$47 + 76 = 40 + 70 + 7 + 6 = 110 + 13 = 123$

Partitioned numbers are then written under one another:

$$\begin{array}{rcrl} 47 & = & 40 + & 7 \\ +\underline{76} & & 70 + & 6 \\ & & \overline{110 +} & 13 = 123 \end{array} \longrightarrow \begin{array}{rl} 40 + & 7 \\ 70 + & 6 \\ \hline 120 + & 3 \\ {\scriptstyle 10} & \end{array}$$

Stage 5: Expanded method in columns

- Move on to a layout showing the addition of the tens to the tens and the ones to the ones separately. To find the partial sums either the tens or the ones can be added first, and the total of the partial sums can be found by adding them in any order. As children gain confidence, ask them to start by adding the ones digits first always.
- The addition of the tens in the calculation 47 + 76 is described in the words 'forty plus seventy equals one hundred and ten', stressing the link to the related fact 'four plus seven equals eleven'.
- The expanded method leads children to the more compact method so that they understand its structure and efficiency. The amount of time that should be spent teaching and practising the expanded method will depend on how secure the children are in their recall of number facts and in their understanding of place value.

Write the numbers in columns.

Adding the tens first:

$$\begin{array}{r} 47 \\ +\ 76 \\ \hline 110 \\ 13 \\ \hline 123 \end{array}$$

Adding the ones first:

$$\begin{array}{r} 47 \\ +\ 76 \\ \hline 13 \\ \underline{110} \\ 123 \end{array}$$

Discuss how adding the ones first gives the same answer as adding the tens first. Refine over time to adding the ones digits first consistently.

$$\begin{array}{rl} 40 + & 7 \\ 70 + & 6 \\ \hline 120 + & 3 \\ {\scriptstyle 10} & \end{array}$$

Stage 6: Column method

- In this method, recording is reduced further. Carry digits are recorded below the line, using a small '1' under the tens column or hundreds column but using the words 'carry ten' (or 'carry one hundred'), not 'carry one'.
- Later, extend to adding three two-digit numbers, two three-digit numbers and numbers with different numbers of digits.

$$\begin{array}{r} 47 \\ +\ 76 \\ \hline \underline{123} \\ {\scriptstyle 1\,1} \end{array} \qquad \begin{array}{r} 258 \\ +\ 87 \\ \hline \underline{345} \\ {\scriptstyle 1\,1} \end{array} \qquad \begin{array}{r} 366 \\ +\underline{458} \\ \underline{824} \\ {\scriptstyle 1\,1} \end{array}$$

Column addition remains efficient when used with larger whole numbers and decimals. Once learned, the method is quick and reliable.

Calculation methods for subtraction

The aim is that children use mental methods when appropriate, but for calculations that they cannot do in their heads they use an efficient written method accurately and with confidence. If, at any time, children find that the method they are using is badly understood and therefore leading to errors, they should return to the last step they did accurately and with understanding. Children are entitled to be taught and to acquire secure mental methods of calculation and an efficient written method of calculation for subtraction which they know they can rely on when mental methods are not appropriate.

It is expected that most children will **begin** to build up to using an efficient method for subtraction of two-digit and three-digit whole numbers by the end of Year 4. (i.e. Stage 5)

To subtract successfully, children need to be able to:

- recall all addition and subtraction facts to 20;
- subtract multiples of 10 (such as 160 – 70) using the related subtraction fact,16 – 7, and their knowledge of place value;
- partition two-digit and three-digit numbers into multiples of one hundred, ten and one in different ways (e.g. partition 74 into 70 + 4 or 60 + 14).

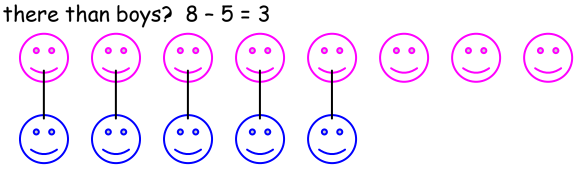
Note: *It is important that children's mental methods of calculation are practised and secured alongside their learning and use of an efficient written method for subtraction.*

<table>
<tr><th colspan="2">The counting-up method of subtraction: A formalised mental method used to illustrate 'Difference' and 'how many more', or for numbers very close together: (to be used in real life applications e.g. for measures and money; and for numbers that are close together. However this method does not aid understanding of the concept of subtraction as 'take away' and 'less than')</th></tr>
<tr><td>

- Using physical apparatus and drawings to match sets and then count on to find the difference.
Note – this is really addition but is recorded as subtraction. It is a useful method of working out an answer to a subtraction problem in real life, but can lead to difficulty if this is taught at the same time as 'subtraction as take away'. To help this misconception, it is often helpful to count up from the smaller group and use the vocabulary 'how many more' and count back from the larger group and use the vocabulary 'how many less' to reinforce the idea of difference as a comparison.

</td><td>

There are 8 girls and 5 boys. <u>How many more</u> girls are there than boys? 8 - 5 = 3

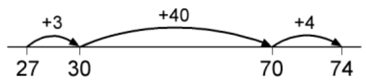


<u>How many less boys</u> are there than girls? 8 - 5 = 3

</td></tr>
<tr><td>

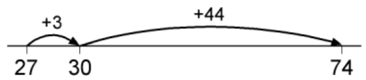
- The <u>mental method</u> of counting up from the smaller to the larger number can be recorded using either number lines or vertically in columns. The number of rows (or steps) can be reduced by combining steps. With two-digit numbers, this requires children to be able to work out the answer to a calculation such as 30 + ☐ = 74 mentally.

</td><td>



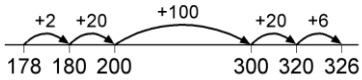
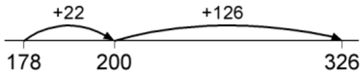
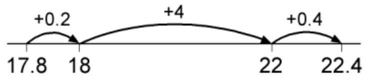
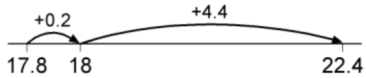
74
– 27
3 → 30
40 → 70
4 → 74
47

or:



74
– 27
3 → 30
44 → 74
47

</td></tr>
</table>

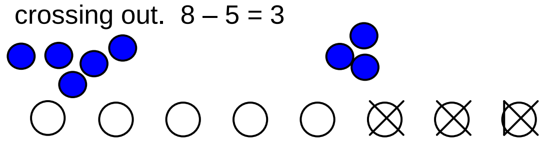
• With three-digit numbers the number of steps can again be reduced, provided that children are able to work out answers to calculations such as 178 + ☐ = 200 and 200 + ☐ = 326 mentally. • The most compact form of recording remains reasonably efficient.	 or:  326 –178 22 → 200 126 → 326 148 326 –178 178 2 → 180 20 → 200 100 → 300 26 → 326 148
! The method can be used with decimals where no more than three columns are required. However, it becomes less efficient when more than three columns are needed. ! This counting-up method can be a useful alternative for children whose progress is slow, whose mental and written calculation skills are weak and whose projected attainment at the end of Key Stage 2 is towards the lower end of level 4 but does not aid progression to more efficient methods in either subtraction or division.	 22.4 –17.8 0.2 → 18 4.0 → 22 0.4 → 22.4 4.6 or:  22.4 –17.8 0.2 → 18 4.4 → 22.4 4.6

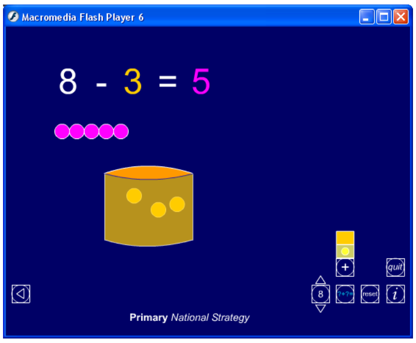
Subtraction as 'Take Away'

Stage 1: Drawing objects and crossing out; counting back

- As addition started with combining groups, subtraction starts with separating groups, or crossing out. It is important that the vocabulary MORE and LESS are introduced to indicate a bigger group or larger number and the remaining smaller group.
- Counting back on a number track, bead string, or just mentally can also be used to reinforce the idea of subtraction resulting in a smaller number or 'Less'. This concept is not supported by the 'counting on' method of subtraction (which is really 'subtraction by addition') so this should usually be demonstrated separately as outlined above.

Apparatus can be used to physically separate a smaller group from a larger and by counting out the smaller group then counting the remaining group, or represented by drawing as crossing out. 8 – 5 = 3

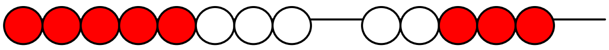


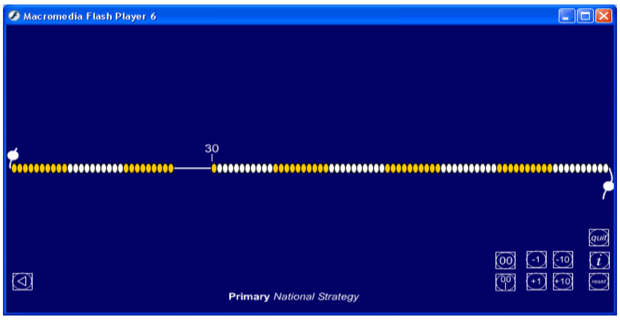


ITP Number Facts

Use a bead strings or bead bars to model subtraction including bridging through ten

There are 13 people on the climbing frame. That is too many people. Five people should get off. How many people should be on the climbing frame?
13 - 5 = 8 (model 13 subtract 3 is 10 then 10 take away 2 makes 8)



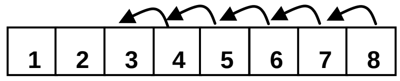


ITP Counting on and Counting back

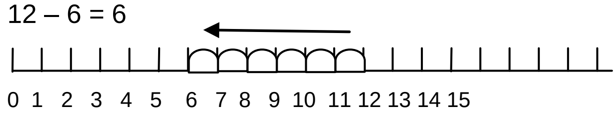
Stage 2: Using a marked and numbered number line

- Use a numbered track to count back (or in outdoor play to jump back/ hop etc)
- Using a fully marked and fully numbered number line to count back
- Use a marked, partially marked or empty number line to **count back** (take away)) and record number sentences.

8 – 5 = 3 Count back 5 (5 **less than** 8)

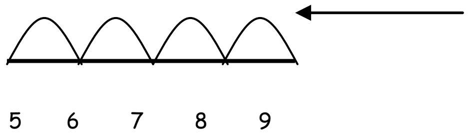


12 – 6 = 6

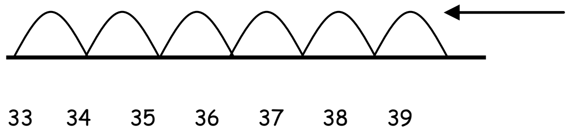


12 - 7 (counting back) - marked line – when multiple of 10 – counting back the answer is the number 'landed' on (5)

1-digit number - 1-digit number e.g. 9 – 4 = 5

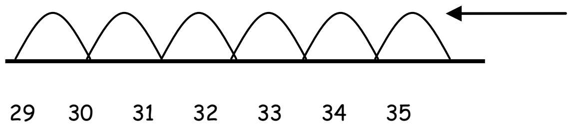


2-digit number - 1-digit number e.g. 39 – 6 = 33
not crossing tens boundary



multiple of 10 - 2-digit number

2-digit number - 1-digit number e.g. 35 – 6 = 33
extend to crossing tens boundary



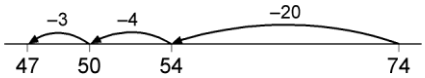
Stage 3: Using the empty number line

- The empty number line helps to record or explain the steps in mental subtraction. A calculation like 74 – 27 can be recorded by counting back 27 from 74 to reach 47. The empty number line is also a useful way of modelling processes such as bridging through a multiple of ten.
- *The steps can also be recorded by counting up from the smaller to the larger number to find the difference, for example by counting up from 27 to 74 in steps totalling 47.*
- With practice, children will need to record less information and decide whether to count back or forward.
- It is useful to ask children whether counting up or back is the more efficient for calculations such as 57 – 12, 86 – 77 or 43 – 28, and whether this is true for all numbers.
- However, in general, counting back will be needed for progression to more efficient standard methods of subtraction and division.

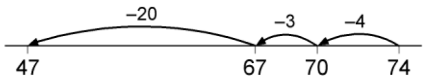
Steps in subtraction can be recorded on a number line. The steps often bridge through a multiple of 10.

15 – 7 = 8

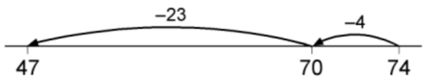
74 – 27 = 47 worked by counting back:



The steps may be recorded in a different order:



or combined:



Stage 4: Partitioning

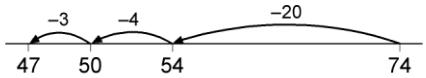
- Subtraction can be recorded using partitioning to write equivalent calculations that can be carried out mentally. For 74 – 27 this involves partitioning the 27 into 20 and 7, and then subtracting from 74 the 20 and the 4 in turn. Some children may need to partition the 74 into 70 + 4 or 60 + 14 to help them carry out the subtraction.

Subtraction can be recorded using partitioning:

74 – 27 = 74 – 20 – 7 = 54 – 7 = 47

74 – 27 = 70 + 4 – 20 – 7 = 60 + 14 – 20 – 7 = 40 + 7

This requires children to subtract a single-digit number or a multiple of 10 from a two-digit number mentally. The method of recording links to counting back on the number line.



Stage 5: Expanded layout, leading to column method

- Partitioning the numbers into tens and ones and writing one under the other mirrors the column method, where ones are placed under ones and tens under tens.
- This does not link directly to mental methods of counting back or up but parallels the partitioning method for addition. It also relies on secure mental skills.
- The expanded method leads children to the more compact method so that they understand its structure and efficiency. The amount of time that should be spent teaching and practising the expanded method will depend on how secure the children are in their recall of number facts and with partitioning.

Partitioned numbers are then written under one another:

Example: 74 − 27

```
   70   4        60  14          6 14
                 ~~70~~ ~~4~~   ~~7~~ ~~4~~
 − 20   7      − 20   7        − 2  7
 ---------     ---------       ------
               40 + 7           4  7
```

Example: 741 − 367

```
                     30   11      600  130  11       6 13 11
  700  40  1     700 ~~40~~ ~~1~~  ~~700~~ ~~40~~ ~~1~~   ~~7~~ ~~4~~ ~~1~~
− 300  60  7   − 300 60   7    − 300   60   7     − 3  6  7
-------------  -------------   ---------------    ---------
                          4     300 + 70 + 4        3  7  4
```


Amended from Renewed Framework Guidance

Stage 6: The expanded method for three-digit numbers

Example: 563 − 241, no adjustment or decomposition needed

Expanded method leading to

	500	60	3
−	200	40	1
	300 +	20 +	2

	563
−	241
	322

Start by subtracting the ones, then the tens, then the hundreds. Refer to subtracting the tens, for example, by saying 'sixty take away forty', not 'six take away four'.

Example: 563 − 271, adjustment from the hundreds to the tens, or partitioning the hundreds

	500	60	3
−	200	70	1

	400	160	
	~~500~~	~~60~~	3
−	200	70	1
	200 +	90 +	2

	400	160	3
−	200	70	1
	200 +	90 +	2

	4	16	
	~~5~~	~~6~~	3
−	2	7	1
	2	9	2

Begin by reading aloud the number from which we are subtracting: 'five hundred and sixty-three'. Then discuss the hundreds, tens and ones components of the number, and how 500 + 60 can be partitioned into 400 + 160. The subtraction of the tens becomes '160 minus 70', an application of subtraction of multiples of ten.

Example: 563 − 278, adjustment from the hundreds to the tens and the tens to the ones

	500	60	3
−	200	70	8

	400	150	13
	~~500~~	~~50~~	13
	~~500~~	~~60~~	~~3~~
−	200	70	8
	200 +	80 +	5

	400	150	13
−	200	70	8
	200 +	80 +	5

	4	15	13
	~~5~~	~~6~~	~~3~~
−	2	7	8
	2	8	5

Here both the tens and the ones digits to be subtracted are bigger than both the tens and the ones digits you are subtracting from. Discuss how 60 + 3 is partitioned into 50 + 13, and then how 500 + 50 can be partitioned into 400 + 150, and how this helps when subtracting.

Example: 503 − 278, dealing with zeros when adjusting

	500	0	3
−	200	70	8

	400	100	
	~~500~~	~~0~~	3
−	200	70	8
	200 +	20 +	5

		90	13
	400	~~100~~	3
	~~500~~	~~0~~	~~3~~
−	200	70	8
	200 +	20 +	5

	400	90	13
−	200	70	8
	200 +	20 +	5

	4	9	13
	~~5~~	~~0~~	~~3~~
−	2	7	8
	2	2	5

Here 0 acts as a place holder for the tens. The adjustment has to be done in two stages. First the 500 + 0 is partitioned into 400 + 100 and then the 100 + 3 is partitioned into 90 + 13.

A number problem such as this, with a 0 as a place holder, may be done more accurately by some pupils as a 'difference' or counting up method. This is one of the special cases where this may be a more useful strategy.

503 – 278

278 + 2 = 280	(+2)	
280 + 20 = 300	(+20)	
300 + 200 = 500	(+200)	
500 + 3 = 503	(+ 3)	Total added on = 200 + 20 + 2 + 3 = 225

Calculation methods for multiplication

The aim is that children use mental methods when appropriate, but for calculations that they cannot do in their heads they use an efficient written method accurately and with confidence.

If, at any time, children find that the method they are using is badly understood and therefore leading to errors, they should return to the last step they did accurately and with understanding.

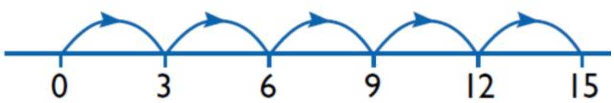
Children are entitled to be taught and to acquire secure mental methods of calculation and an efficient written method of calculation for multiplication which they know they can rely on when mental methods are not appropriate.

These notes show the stages in building up to using an efficient method for two-digit by one-digit multiplication by the end of Year 4, two-digit by two-digit multiplication by the end of Year 5, and three-digit by two-digit multiplication by the end of Year 6.

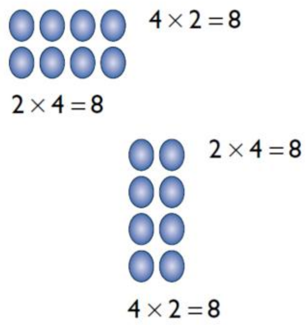
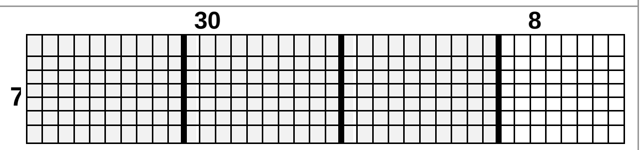
To multiply successfully, children need to be able to:

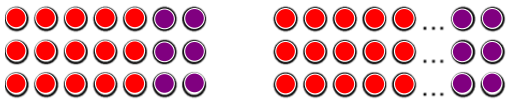
- recall all multiplication facts to 10 × 10;
- partition number into multiples of one hundred, ten and one;
- work out products such as 70 × 5, 70 × 50, 700 × 5 or 700 × 50 using the related fact 7 × 5 and their knowledge of place value;
- add two or more single-digit numbers mentally;
- add multiples of 10 (such as 60 + 70) or of 100 (such as 600 + 700) using the related addition fact, 6 + 7, and their knowledge of place value;
- add combinations of whole numbers using the column method (see above).

Note: *It is important that children's mental methods of calculation are practised and secured alongside their learning and use of an efficient written method for multiplication.*

Stage 1: Continual addition of groups/steps the same size	
• Counting groups / sets • Counting in steps on a number line	 2 + 2 + 2 + 2 + 2 5 lots of 2 ⟶ 5 x 2 NB The use of the term 'lots of' for multiplication is very important and will help children understand terms later such as ¾ of (lots of) 75 meaning ¾ x 75  5 'jumps' of 3.. 5 lots of 3 ⟶ 5 x 3

2 + 2 + 2 +
2 × 5 = 10
2 multiplied
5 pairs

Stage 2: Arrays / leading to Area Method	
• Starting with simple arrangements within times tables knowledge – to show groups (now ordered into rows or columns of equal size.. link to area!), moving to larger numbers. Keep the terminology 'lots of' for multiplication i.e. an array of 8 can be arranged as 2 lots of 4 or 4 lots of 2	
• The area method is useful for showing how the distributive law (needed for the following partitioning and grid methods) works. 7	 **7 lots of 38 is the same as 7 lots of 30 + 7 lots of 8** **38 x 7 = (30 + 8) x 7** **= (30 x 7) + (8 x 7)** **= 210 + 56**

Stage 3: Mental multiplication using partitioning	
• Mental methods for multiplying TU × U can follow on from the area method, mentally (or using jottings) on the distributive law of multiplication over addition. This allows the tens and ones to be multiplied separately to form partial products. These are then added to find the total product. Either the tens or the ones can be multiplied first but it is more common to start with the tens.	Record mental multiplication using partitioning : $14 \times 3 = (10 + 4) \times 3$ $= (10 \times 3) + (4 \times 3) = 30 + 12 = 42$ $43 \times 6 = (40 + 3) \times 6$ $= (40 \times 6) + (3 \times 6) = 240 + 18 = 258$ Note: Use simpler numbers when first using mentally without the area method… revert to area method when children are uncertain about partitioning and distributive law. Children should be introduced to the principle of this law (not its name) in Years 2 and 3, for example when they use their knowledge of the 2, 5 and 10 times-tables to work out multiples of 7 and other tables not yet learnt by rote.  7 x 3 = (5 x 3) + (2 x 3) = 15 + 6 = 21

Stage 4: The grid method	
• The area method leads into an expanded method which uses a grid. This is based on the distributive law and links directly to the mental method. It is an alternative way of recording the same steps.	38 × 7 = (30 × 7) + (8 × 7) = 210 + 56 = 266 x | 30 | 8 7 | 210 | 56 → 266

Amended from Renewed Framework Guidance

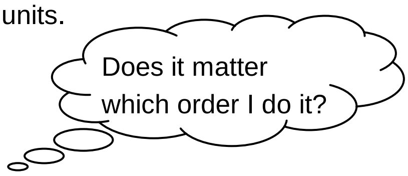
Stage 5: Expanded short multiplication

- The next step is to represent the method of recording in a column format, but showing the working. Draw attention to the links with the area and grid method above.
- Children should describe what they do by referring to the actual values of the digits in the columns. For example, the first step in 38 × 7 is 'thirty multiplied by seven', not 'three times seven', although the relationship 3 × 7 should be stressed.
- Renewed framework expectation is that most children should be able to use this expanded method for TU × U by the end of Year 4.

$$\begin{array}{r} 30+8 \\ \times\ \underline{\quad 7} \\ 210 \\ \underline{56} \\ \underline{266} \end{array} \qquad \begin{array}{l} \\ \\ 30\times7=210 \\ 8\times7=\ \ 56 \\ \\ \end{array} \qquad \begin{array}{r} 38 \\ \times\ \underline{\quad 7} \\ 210 \\ \underline{56} \\ \underline{266} \end{array}$$

Starting from tens to match the grid method. Then move into units first... using place value to simplify the calculations ("7 eights, 7 thirties") leading to standard efficient method which starts by multiplying units.

$$\begin{array}{r} 38 \\ \text{X}\ \underline{\quad 7} \\ 56 \\ \underline{210} \\ \underline{266} \end{array}$$



Stage 6: Short multiplication

- The recording is reduced further, with carry digits recorded below the line.
- If, after practice, children cannot use the compact method without making errors, they should return to the last method that they could use confidently and accurately. This will often be the grid method which works well for decimals too.

$$\begin{array}{r} 38 \\ \times\ \underline{\quad 7} \\ \underline{266} \\ {\scriptstyle 5} \end{array}$$

The step here involves adding 210 and 50 mentally with only the 5 in the 50 recorded. This highlights the need for children to be able to add a multiple of 10 to a two-digit or three-digit number mentally before they reach this stage.

Stage 7: Two-digit by two-digit products

- Extend to TU × TU, asking children to estimate first.
- Start with the grid method. The partial products in each row are added, and then the two sums at the end of each row are added to find the total product.

56 × 27 is approximately 60 × 30 = 1800.

x	50	6		
20	1000	120	1120	1000 + 120
7	350	42	392	350 + 42
			1512	

- Reduce the recording, showing the links to the grid method above.

56 × 27 is approximately 60 × 30 = 1800.

$$\begin{array}{r} 56 \\ \times\ \underline{27} \\ 1000 \\ 120 \\ 350 \\ \underline{42} \\ \underline{1512} \\ {\scriptstyle 1\ \ \ } \end{array} \qquad \begin{array}{r} \\ \\ 50\times20=1000 \\ 6\times20=\ \ 120 \\ 50\times7=\ \ 350 \\ 6\times7=\ \ \ \ 42 \\ \\ \\ \end{array}$$

- Reduce the recording further.
- The carry digits in the partial products of 56 × 20 = 120 and 56 × 7 = 392 are usually carried mentally, but if they are recorded, this should be small and out of the way of the main calculation.
- Renewed framework expectation is for most children to use this long multiplication method for TU × TU by the end of Year 5.

56 × 27 is approximately 60 × 30 = 1800.

$$\begin{array}{r} 56 \\ \times\ \underline{27} \\ 1120 \\ \underline{392} \\ \underline{1512} \\ {\scriptstyle 1\ \ \ } \end{array} \qquad \begin{array}{l} \\ \\ 56\times20 \\ 56\times\ 7 \\ \\ \\ \end{array}$$

Stage 8: Three-digit by two-digit products

- Extend to HTU × TU asking children to estimate first. Start with the grid method.
- It is better to place the number with the most digits in the left-hand column of the grid so that it is easier to add the partial products.
- It is important that children get into the habit of thinking in multiples of ten to ensure they apply their number facts accurately.
- Start with the grid method. The partial products in each row are added, and then the two sums at the end of each row are added to find the total product.
- The grid method may be the main method used by children whose progress is slow, whose mental and written calculation skills are weak and whose projected attainment at the end of Key Stage 2 is towards the lower end of level 4.

538 x 73 is approximately 500 x 70 = 35 000

x	500	30	8		
70	35000 (35 x10x10x10)	2100	560	37660	35000 + 2100 + 560
3	1500	90	24	1614	1500 + 90 + 24
				39274	

- Reduce the recording, showing the links to the grid method above.
- This expanded method is cumbersome, with six multiplications and a lengthy addition of numbers with different numbers of digits to be carried out. There is plenty of incentive to move on to a more efficient method.

```
  538
x  73
35000   500 x 70
 2100    30 x 70
  560     8 x 70
 1500   500 x 3
   90    30 x 3
   24     8 x 3
39274
 1 1
```

- Children who are already secure with multiplication for TU × U and TU × TU should have little difficulty in using the same method for HTU × TU.
- Again, the carry digits in the partial products are usually carried mentally.

! Again, the carry digits in the partial products are usually carried mentally, although if children need to jot these down, they should be modelled in such a way to prevent them becoming confused with the digits to be added.

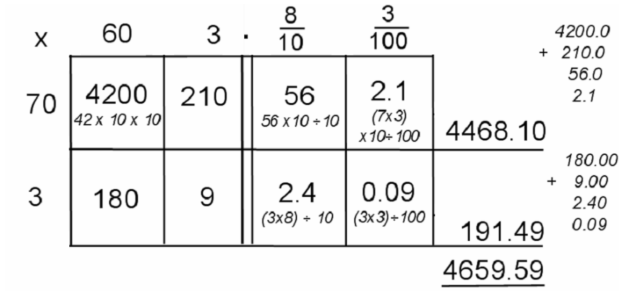
538 x 73 is approximately 500 x 70 = 35 000

```
  538
x  73
37660   538 x 70
 1614   538 x  3
39274
 1
```

Stage 9: Extending the grid method to decimals

- Ask children to estimate first,
- For decimals remind children of the place value f large and small numbers i.e. that numbers with a zero are multiples (x10) of ten, and that decimals are ÷ 10, ÷ 100 etc respectively
- The grid method may be the main method used by children whose progress is slow, whose mental and written calculation skills are weak and whose projected attainment at the end of Key Stage 2 is towards the lower end of level 4.

63.83 x 73 is approximately 60 x 70 = 4200



Calculation methods for division

The aim is that children use mental methods when appropriate, but for calculations that they cannot do in their heads they use an efficient written method accurately and with confidence.

If, at any time, children find that the method they are using is badly understood and therefore leading to errors, they should return to the last step they did accurately and with understanding.

Children are entitled to be taught and to acquire secure mental methods of calculation and an efficient written method of calculation for division which they know they can rely on when mental methods are not appropriate.

These notes show the stages in building up to long division through Years 4 to 6 – first long division TU ÷ U, extending to HTU ÷ U, then HTU ÷ TU, and then short division HTU ÷ U.

To divide successfully in their heads, children need to be able to:

- understand and use the vocabulary of division – for example in 18 ÷ 3 = 6, the 18 is the dividend, the 3 is the divisor and the 6 is the quotient;
- partition two-digit and three-digit numbers into multiples of 100, 10 and 1 in different ways;
- recall multiplication and division facts to 10 × 10, recognise multiples of one-digit numbers and divide multiples of 10 or 100 by a single-digit number using their knowledge of division facts and place value;
- know how to find a remainder working mentally – for example, find the remainder when 48 is divided by 5;
- understand and use multiplication and division as inverse operations.

Note: *It is important that children's mental methods of calculation are practised and secured alongside their learning and use of an efficient written method for division.*

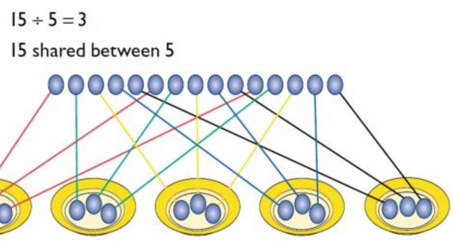
To carry out written methods of division successful, children also need to be able to:

- understand division as repeated subtraction;
- estimate how many times one number divides into another – for example, how many sixes there are in 47, or how many 23s there are in 92;
- multiply a two-digit number by a single-digit number mentally;
- subtract numbers using the column method.

Stage 1: Sharing

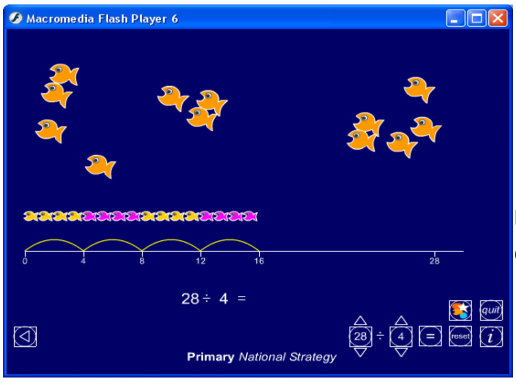
- Sharing is an early step to division but not efficient for larger numbers

One for me… one for you…



Stage 2: Grouping or continual subtraction

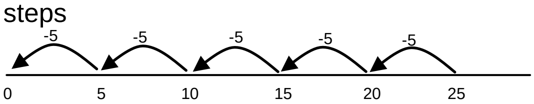
- This is necessary precursor to ‘chunking’ and more efficient for larger numbers. It also helps to show that division is the inverse of multiplication (Grouping)



NNS ITP

Grouping

Jumping back on a number line in same size steps

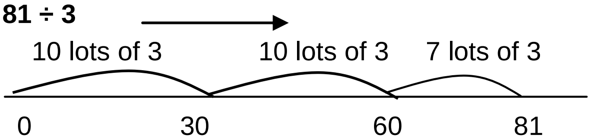


Continual subtraction of the same steps (number line or apparatus)

Stage 3: Division of TU ÷ U on a number line (beginning ‘chunking’)

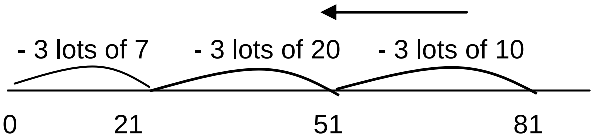
- The process of continual subtraction can be extended, increasing the size of the groups to take away... e.g. if I am taking away 10 equal groups of 4, I may as well do it all at once – as one group of 40! This increasing efficiency of subtracting groups is known as ‘chunking’.

81 ÷ 3



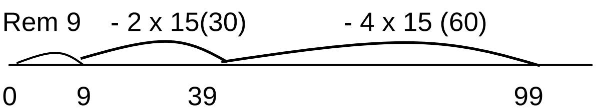
Use tables knowledge to fit ‘chunks’ of (or groups of) 10 into the number, until you can’t take another lot of 10. Use tables knowledge to work out the next best ‘chunk’. This early step helps children to prepare for the next step.

‘Chunking’ backwards (continued subtraction) helps to prepare for more efficient methods of division.



- This method can be extended to include TU ÷ TU

99 ÷ 15



99 ÷15 = (4 + 2) Remainder 9 = 6 r 9

Stage 4: Short division of TU ÷ U

- Short division of a two-digit number can be introduced to children who are confident with multiplication and division facts and with subtracting multiples of 10 mentally, and whose understanding of partitioning and place value is sound.
- Renewed framework expectation for the end of year 4 is that children will develop and use written methods to record, support and explain division of two-digit numbers by a one-digit number, including division with remainders (e.g. 15 × 9, 98 ÷ 6)

81 ÷ 3

What number facts can we apply to help us solve this?

3 x 10 = 30.

3 x 20 = 60

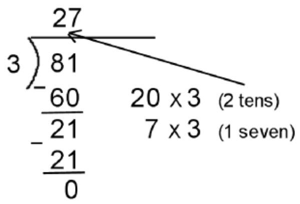
3 x 30 = 90 (Too much!) Answer will be more than 20 and less than 30

81 – 60 = 21. (20 3s)

3 x 7 = 21 (7 3s)

So 81 – 60 – 27 = 0

We took away 60 – that's 20 '3s' and 21 that's 7 '3s' so altogether we took away 27 '3's



Stage 5: Written recording of chunking for TU ÷ U

- The number line method of division, with more efficient groups being subtracted, can be recorded vertically.

81 ÷ 3

```
   81
 - 30  ( 10 lots of 3)
   51
 - 30  (10 lots of 3)
   21
 - 21 ( 7 lots of 3)
    0
```

So, 81 = 10 lots of 3 + 10 lots of 3 + 7 lots of 3

=10+10+7 lots of 3

= 27

Or

81÷ 3 = 10 + 10 + 7

= 27

Stage 6: Written chunking (expanded method) for TU÷T and HTU ÷T

- This method is based on subtracting multiples of the divisor from the number to be divided, the dividend.
- For TU ÷ U there is a link to the mental method.
- As you record the division, ask: ‘How many nines in 90?’ or ‘What is 90 divided by 9?’
- Once they understand and can apply the method, children should be able to move on from TU ÷ U to HTU ÷ U quite quickly as the principles are the same.
- This method, often referred to as ‘chunking’, is based on subtracting multiples of the divisor, or ‘chunks’. Initially children subtract several chunks, but with practice they should look for the biggest multiples of the divisor that they can find to subtract.
- Chunking is useful for reminding children of the link between division and repeated subtraction.
- However, children need to recognise that chunking is inefficient if too many subtractions have to be carried out. Encourage them to reduce the number of steps and move them on quickly to finding the largest possible multiples.

97 ÷ 9

```
  9)97
  − 90    9 × 10
     7
Answer:   10 R 7
```

```
  6)196
  −  60    6 × 10
     136
  −  60    6 × 10
      76
  −  60    6 × 10
      16
  −  12    6 ×  2
       4      32
Answer:     32 R 4
```

Stage 7: ‘Expanded’ method for HTU ÷ U (Bigger chunks)

- The key to the efficiency of chunking lies in the estimate that is made before the chunking starts. Estimating for HTU ÷ U involves multiplying the divisor by multiples of 10 to find the two multiples that ‘trap’ the HTU dividend.
- Estimating has two purposes when doing a division:
  - to help to choose a starting point for the division;
  - to check the answer after the calculation.
- Children who have a secure knowledge of multiplication facts and place value should be able to move on quickly to the more efficient recording on the right.

To find 196 ÷ 6, we start by multiplying 6 by 10, 20, 30, … to find that 6 × 30 = 180 and 6 × 40 = 240. The multiples of 180 and 240 trap the number 196. This tells us that the answer to 196 ÷ 6 is between 30 and 40.

Start the division by first subtracting 180, leaving 16, and then subtracting the largest possible multiple of 6, which is 12, leaving 4.

```
  6)196
  − 180    6 × 30
      16
  −  12    6 ×  2
       4      32
Answer:     32 R 4
```

The quotient 32 (with a remainder of 4) lies between 30 and 40, as predicted.

Stage 8: Short division of HTU ÷ U

- Short division is an increasingly efficient method of recording division
- The accompanying patter is 'How many threes in 290?' (the answer must be a multiple of 10). The closest you can get is 270 which is 90 threes (9 tens), so this must be placed in the 'tens' column.
- This leaves 21 remaining. We now ask: 'How many threes in 21?' which has the answer 7 (t 'ones' so this must be placed in the 'ones' column).
- Short division of a three-digit number can be introduced to children who are confident with multiplication and division facts and with subtracting multiples of 10 mentally, and whose understanding of partitioning and place value is sound.

```
     9 7
3 ) 2 9 ²1
```

The carry digit '2' represents the 2 tens (20 remaining) that have been exchanged for 20 ones. In the first recording above it is written in front of the 1 to show that a total of 21 ones are to be divided by 3.

The 97 written above the line represents the answer: 90 + 7, or 9 tens and 7 ones.

Stage 9: Long division of HTU ÷ TU

- The next step is to tackle HTU ÷ TU, which for most children will be in Year 6.
- The layout on the right, which links to chunking, is in essence the 'long division' method.
- How many groups of 24 are there in 560? The closest we can get is 480 which is 24 x 20 or 24 x 2 tens
- So the 2 (tens) is often recorded above the 'bus stop' in the tens column.
- Take 24 groups of 20 away from 560 leaves 80 How many groups of 24 are there in the remaining 80? The closest we can get is 72 which is 24 x 3 with a remainder of 8.
- The 3 ones are often recorded above the 'bus stop' in the ones column.
- This make 20 + 3 (often recorded more efficiently as 2 (tens) and 3 (ones) without needing the HTU exemplification.

How many packs of 24 can we make from 560 biscuits? Start by multiplying 24 by multiples of 10 to get an estimate. As 24 × 20 = 480 and 24 × 30 = 720, we know the answer lies between 20 and 30 packs. We start by subtracting 480 from 560.

```
      H T U
        2 0
          3
24 )  560
    − 480     24 × 20
       80     (24 x 20 = 24 x 2 tens)
       72     24 × 3
        8
```

Answer: 23 R 8

In effect, the recording above is the long division method, though conventionally the digits of the answer are recorded above the line as shown below.

```
        23
24 )   560
      −480
        80
       −72
         8
```

Answer: 23 R 8

Stage 10: Short division of HTU ÷ TU

- For children who are really confident in all the above, the following short method may be appropriate; but if children begin to make mistakes, they should go back to the last method they could use with consistent speed and accuracy

```
       2 3 r 8
24 ) 5 6 ⁸0
```

Think in tens, then in ones
2 x 24 tens = 48 tens (put answer into tens column)
56 tens - 48 tens = 8 tens
That's 80 ones.. are there any other ones to add to this? 3 x 24 = 72 80 (ones) – 72 (ones) = 8

That's a remainder of 8, or 8 ÷ 24 or $\frac{8}{24}$